## *Lecture T1*

# *Transient Mass Transfer*

Up to now, we have considered either processes applied to closed systems or processes involving steady-state flows. In this lecture we turn our attention to transient mass-transfer situations. Recall that the general material balance takes the form

$$\begin{pmatrix}\text{Rate of material}\\ \text{into system}\end{pmatrix} - \begin{pmatrix}\text{Rate of material}\\ \text{out of system}\end{pmatrix} = \begin{pmatrix}\text{Rate of accumulation}\\ \text{of material in system}\end{pmatrix} \tag{T1.1}$$

When the rhs = 0, the process is a steady-state mass transfer; when rhs ≠ 0, the process is a transient. If rhs > 0, then the amount of material in the system is increasing, while rhs < 0 indicates the amount is decreasing.

Transient situations are best discussed in terms of examples; in this lecture the examples involve fluids discharged from tanks. In these situations we can use thermodynamics to describe energy losses to friction as fluids flow through pipes, fittings, and orifices. Frictional losses occur at system boundaries, so we first consider how changes in the boundary energy contribute to the general energy balance (§ T1.1). Then we present examples (§ T1.2 and T1.3).

### *T1.1 Energy of the Boundary*

In previous lectures, whenever we used the steady-state energy balance (such as in (24.8)), we tacitly ignored changes in the energy of the boundary $E_b$. In many applications this is a good assumption, for $\Delta E_b$ is often negligible compared to changes in the system's internal and external energies. But $E_b$ changes because of friction caused by fluids flowing through pipes and fittings; so, if we want to consider such frictional effects, we must include $\Delta E_b$ in the energy balance. Then the steady-state energy balance looks like this:

$$\Delta E_k + \Delta E_p + \Delta H + \Delta E_{\mathrm{b}} = Q + W_s \tag{T1.2}$$

In general, $E_{\mathrm{b}}$ can change because of interactions between the boundary and the surroundings or between the boundary and the system. Here we are considering changes in $E_{\mathrm{b}}$ caused by frictional forces created by the flowing fluid; in these situations, friction always tends to increase $E_{\mathrm{b}}$.

***Problem Statement.*** Figure T1.1 shows a gravity feed tank that empties into an open drain through a horizontal pipe run. Flow is controlled by a valve in the horizontal pipe. In this section we consider only steady flow, which is maintained by a make-up feed to the top of the tank. Thus, the water level is constant at the height $h$. A similar situation was analyzed in § 24.2, but there we ignored frictional losses. We seek an expression for the outlet velocity at 3 in terms of the head $h$ with the effects of friction included.

***Picture.*** Figure T1.1

***System.*** Water in horizontal pipe between stations 2 and 3.

***Always True.*** Steady-state material balance

$$m_2 = m_3 \tag{T1.3}$$

and steady-state energy balance (T1.2).

***Model.*** We need a model for $\Delta E_{\mathrm{b}}$. We assume all of $\Delta E_{\mathrm{b}}$ can be attributed to friction at the pipe and valve walls. For turbulent flow through pipes and valves, experiment shows that such frictional losses are proportional to the

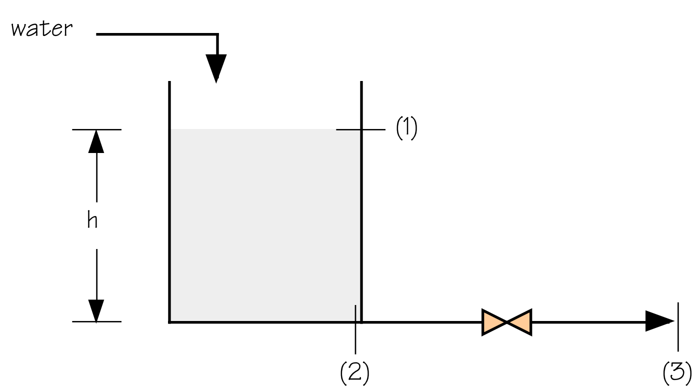



***Figure T1.1*** Schematic of a gravity feed tank supplying water, at steady state, through a horizontal pipe to an open drain. Note $h \geq 0$.

corresponding pressure drop. In the present case, we assume $\Delta E_b$ between stations 2 and 3 is proportional to the pressure drop $\Delta P_{23}$,

$$\Delta E_b \propto -\Delta P_{23} \tag{T1.4}$$

Further, since stations 1 and 3 are each open to the atmosphere,

$$\Delta P_{23} = -\Delta P_{12} \tag{T1.5}$$

and $\Delta P_{12}$ is caused by the head in the tank,

$$\frac{\Delta P_{12}}{\rho} = \frac{mgh}{g_c} \tag{T1.6}$$

Thus as a model, we combine (T1.4)-(T1.6) and assume the frictional losses to the boundary are some fraction $C$ of the head,

$$\Delta E_b = C\frac{mgh}{g_c} \tag{T1.7}$$

where $C < 1$. The fraction $C$ is a process variable whose value must be obtained from experiment or from a correlation. Its value will depend on the length, diameter, and material of the pipe, on the type of valve, and on the valve setting—that is, on how far we have opened the valve.

***Strategy.*** Between stations 2 and 3 along the horizontal pipe in Figure T1.1, we have $W_s = 0$, $Q = 0$, $\Delta E_p = 0$, and $\Delta U = 0$, so $\Delta H = \Delta P/\rho$. Then the energy balance (T1.2) simplifies to

$$\Delta E_k + \frac{\Delta P_{23}}{\rho} = -C\frac{mgh}{g_c} \tag{T1.8}$$

Combining (T1.5) and (T1.6) with (T1.8) leaves

$$\frac{m\Delta\upsilon^2}{2g_c} = (1-C)\frac{mgh}{g_c} \tag{T1.9}$$

where $h \geq 0$. This equation states that part of the head in the feed tank is used to change the kinetic energy of the water, while the rest is used to overcome frictional resistance at the pipe and valve walls.

At the bottom of the tank, the velocity is zero; $\upsilon_2 = 0$. Then we can solve (T1.9) for the exit velocity $\upsilon$. The result is

$$\upsilon = \sqrt{2(1-C)gh} \tag{T1.10}$$

Thus, the exit velocity is proportional to the root of the head,

$$\upsilon = k\sqrt{h} \tag{T1.11}$$

where

$$k = \sqrt{2(1-C)g} \tag{T1.12}$$

Equation (T1.11) is the desired result; it gives the steady-state exit velocity from a gravity feed tank in terms of the height of fluid in the tank. We caution that (T1.11) is a model result, based on the presumption (T1.7). Physically, the constant $k$ can be interpreted as a conductance or a reciprocal resistance: large values of $k$ correspond to small resistances ($C$ small) and therefore high exit velocities.

## *T1.2 Transient Process: Tank Draining*

We convert the steady-state process in Figure T1.1 into a transient by stopping the flow of make-up water to the top of the tank. Then the situation appears as in Figure T1.2. The frictional resistance offered by the pipe and valve can be estimated by observing the rate at which the water level starts to fall; for example, we consider the situation in which the level falls by 10% during the first minute of transient flow. That is, if we let $h_o$ represent the initial height of water in the tank, then

$$h = 0.9h_o \qquad \text{at } t = 1 \text{ min} \tag{T1.13}$$

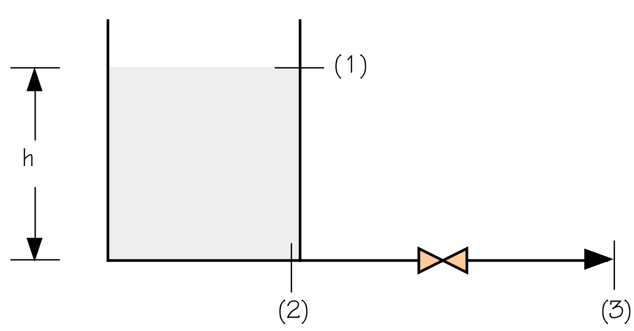



***Figure T1.2*** Schematic of a gravity feed tank emptying through a horizontal pipe to an open drain.

This change could be observed via a sight glass on the side of the tank. A typical problem is to find the time needed for the tank to completely drain.

***Picture.*** Figure T1.2

***System.*** Water in horizontal pipe between stations 2 and 3.

***Always True.*** General material balance (T1.1).

***Model.*** We use the model (T1.7) for the frictional losses to the pipe walls and valve body. This means we are assuming that the steady-state model, rationalized in § T1.1, applies to transient situations.

***Strategy.*** The process in Figure T1.2 has one outlet and no inlets, so the general material balance (T1.1) reduces to

$$0 - \begin{pmatrix}\text{Rate of material} \\ \text{out of system}\end{pmatrix} = \begin{pmatrix}\text{Rate of accumulation} \\ \text{of material in system}\end{pmatrix} \tag{T1.14}$$

We first obtain a general form for the solution (§ T1.2.1) and then evaluate a particular solution using specific numerical values (§ T1.2.2.).

### *T1.2.1 General Solution*

Let $V$ be the volume of water in the tank, $\rho$ be the density of water, and $\mathcal{A}_p$ be the cross-sectional area of the pipe at station 3. Then the material balance (T1.14) can be written as

$$-\rho \upsilon \mathcal{A}_p = \rho \frac{dV}{dt} \tag{T1.15}$$

where $\upsilon$ is the linear velocity at the exit, station 3. Let $\mathcal{A}_t$ be the cross-sectional area of the tank and use the model result (T1.11) to relate the velocity $\upsilon$ to the head $h$. Then (T1.15) becomes

$$-\mathcal{A}_p k \sqrt{h} = \mathcal{A}_t \frac{dh}{dt} \tag{T1.16}$$

Define $\beta$ to be the ratio of pipe diameter to tank diameter, $\beta = d_p / d_t$, so

$$\beta^2 = \frac{\mathcal{A}_p}{\mathcal{A}_t} \tag{T1.17}$$

Using this in (T1.16) and separating variables leaves

$$\frac{dh}{\sqrt{h}} = -k\beta^2 dt \tag{T1.18}$$

We integrate this from the initial condition ($h = h_o$ at $t = 0$) to any subsequent time, at which $h = h(t)$. The result is

$$\sqrt{h} = \sqrt{h_o} - \frac{1}{2} k\beta^2 t \tag{T1.19}$$

This gives the height of water in the tank at any time $t$ during the transient. The time for the tank to completely empty, $t_e$, is given by (T1.19) with $h = 0$,

$$t_e = \frac{2\sqrt{h_o}}{k\beta^2} \tag{T1.20}$$

Thus, small values of the frictional resistance correspond to the large values of the conductance $k$, which result in a short time to empty the tank.

### T1.2.2 Particular Solution

To obtain a value for the conductance $k$, we use the experimental observation (T1.13). Thus, substituting (T1.13) into the general solution (T1.19) yields

$$k\beta^2 = 0.103\sqrt{h_o} \tag{T1.21}$$

If $h_o$ is in feet, then (T1.21) gives $k\beta^2$ in ($ft^{1/2}$/min). Combining this with (T1.19) produces the particular solution

$$h = (1 - 0.051t)^2 h_o \tag{T1.22}$$

The head decreases as the square of the elapsed time. Substituting (T1.21) into (T1.20) gives

$$t_e = 19.5 \text{ min} \tag{T1.23}$$

as the total time needed to empty the tank. This results depends implicitly on the initial head $h_o$ and the pipe resistance through the condition (T1.13).

As the tank drains, the exit velocity $\upsilon$ decreases. To find how $\upsilon$ changes with time, substitute (T1.22) into the model (T1.11) to find

$$\upsilon = k(1 - 0.051t)\sqrt{h_o} \qquad \text{(T1.24)}$$

Thus, the exit velocity decreases linearly as the tank drains.

Finally, we can estimate the fraction of the head that is used to overcome the friction in the pipe and valve. This is obtained by combining (T1.21) with the model (T1.12),

$$k = \sqrt{2(1-C)g} = 0.103\beta^{-2}\sqrt{h_o}\ (\sqrt{\text{ft}}/\text{min}) \qquad \text{(T1.25)}$$

For known values of the pipe diameters ($\beta$) and initial height ($h_o$), we can solve (T1.25) for the fraction $C$.

To have a particular numerical example, say the initial head is $h_o = 10$ ft, the pipe diameter is $d_p = 2$ in, and the tank diameter is $d_t = 5$ ft. Then

$$\beta^2 = \frac{1}{900} \qquad \text{(T1.26)}$$

The resulting changes in the head $h(t)$ and exit velocity $\upsilon(t)$ are shown in Figure T1.3. The fraction of the head used to overcome friction is given by (T1.25) as

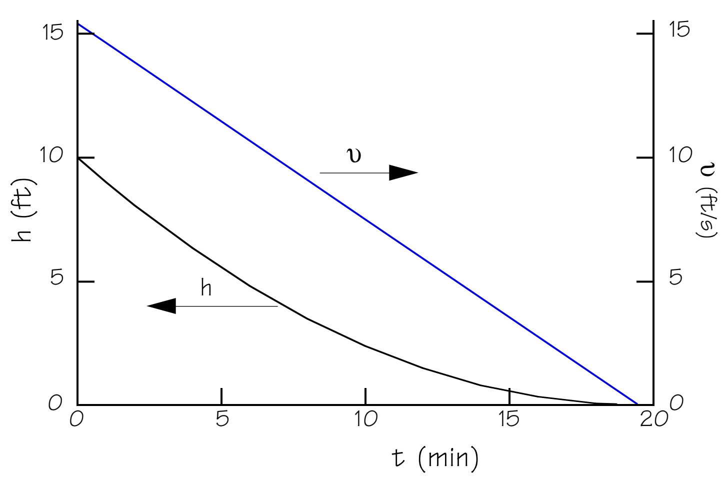



***Figure T1.3*** For a tank draining, as in Figure T1.2, the exit velocity of water decreases linearly with time (T1.24), but height of water in tank decreases as the square of $t$ (T1.22).

$$C = 0.63 \tag{T1.27}$$

That is, nearly two-thirds of the head is lost to friction at the pipe walls and in the valve body. Recall, this fraction is controlled, to some extent, by the position of the valve.

## T1.3 Gas-Tank Rupture

As another example, we consider a sealed tank charged with air to 10 bar at 20°C. The tank has a 2m diameter and is 3m high. It sits in a room that is at 1 bar and 20°C. Suddenly, the tank ruptures, creating a 0.25-in diameter hole in the tank wall. After 15 seconds, the tank pressure has fallen to 7.75 bar. How long does it take for the tank pressure to reach 1 bar?

The picture is shown in Figure T1.4; the system is the air in the tank. The always true is the general material balance (T1.1). For the substance model, we use the ideal gas. To model the frictional resistance posed by the hole, we use the same analysis as in § T1.1. In particular, the hole is an orifice, and (from a Bernoulli equation) the linear velocity through an orifice is proportional to the root of the pressure drop,

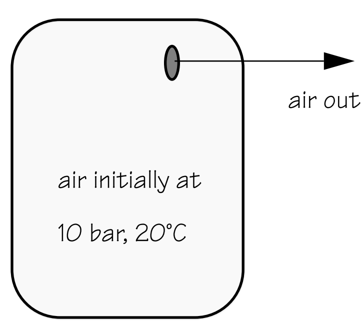
$$\upsilon = k\sqrt{\Delta P} \tag{T1.28}$$

The constant $k$ is proportional to an orifice coefficient.

Let $V$ be the volume of the tank, $\rho$ be the density of air in the tank, and $\mathcal{A}$ be the cross-sectional area of the hole. Then the general material balance for this situation can be written as

$$\frac{d(V\rho)}{dt} = -\rho\upsilon\mathcal{A} \tag{T1.29}$$

***Figure T1.4*** A rigid tank containing air at 10 bar, 20°C fails, allowing air to escape through a 0.25-in diameter hole. How long until the pressure reaches ambient?



Substituting (T1.28) into (T1.29) and noting that the tank volume is constant, we have

$$V\frac{d\rho}{dt} = -\rho \mathcal{A} k\sqrt{\Delta P} \tag{T1.30}$$

We can use the ideal-gas law to write the pressure drop across the orifice in terms of the difference in densities between the air inside and outside the tank,

$$V\frac{d\rho}{dt} = -\rho \mathcal{A} k\sqrt{RT}\sqrt{\Delta\rho} \tag{T1.31}$$

There is no Joule-Thomson effect for ideal gases, so the temperature remains constant.

Collecting all constants in (T1.31) into a single term, we write

$$\frac{d\rho}{dt} = -G\rho\sqrt{(\rho - \rho_a)} \tag{T1.32}$$

where $\rho_a$ is the density of air in the room and $G$ is the constant,

$$G = \frac{\mathcal{A} k\sqrt{RT}}{V} \tag{T1.33}$$

Separating variables in (T1.32), we have

$$\frac{d\rho}{\rho\sqrt{(\rho - \rho_a)}} = -\,G dt \tag{T1.34}$$

We use integral tables to evaluate the lhs; the integration is done from $t = 0$, where the density takes its initial value $\rho_o$, to any subsequent $t$.

$$\frac{2}{\sqrt{\rho_a}}\left[\tan^{-1}\left(\frac{\rho - \rho_a}{\rho_a}\right) - \tan^{-1}\left(\frac{\rho_o - \rho_a}{\rho_a}\right)\right] = -Gt \tag{T1.35}$$

The arguments of the inverse tangent must be in radians. We can use the gas law to write this in terms of the pressures,

$$\frac{2}{\sqrt{P_a/(RT)}}\left[\tan^{-1}\left(\frac{P-P_a}{P_a}\right)-\tan^{-1}\left(\frac{P_o-P_a}{P_a}\right)\right] = -Gt \tag{T1.36}$$

This is the general solution; it gives the pressure $P$ in the tank at any time $t$ after the failure. We can find the constant $G$ by using the given observation that $P = 7.75$ bar when $t = 0.25$ min. Thus,

$$\frac{G\sqrt{P_a/(RT)}}{2} = \frac{-1}{0.25\text{ min}}\left[\tan^{-1}\left(\frac{7.75-1}{1}\right)-\tan^{-1}\left(\frac{10-1}{1}\right)\right] \tag{T1.37}$$

$$= 0.146\ /\text{min} \tag{T1.38}$$

We can now solve (T1.36) for the time $t_e$ needed for the tank to reach $P = 1$ bar,

$$t = \frac{\tan^{-1}(9)}{0.146} = 10\text{ min} \tag{T1.39}$$

The complete profile for the time dependence of the pressure $P$ in the tank is shown in Figure T1.5. Although this analysis is analogous to that in § T1.2, the two differ because here the gas density changes during the process, while in § T1.2 the liquid density remained constant. This led to different differential equations.

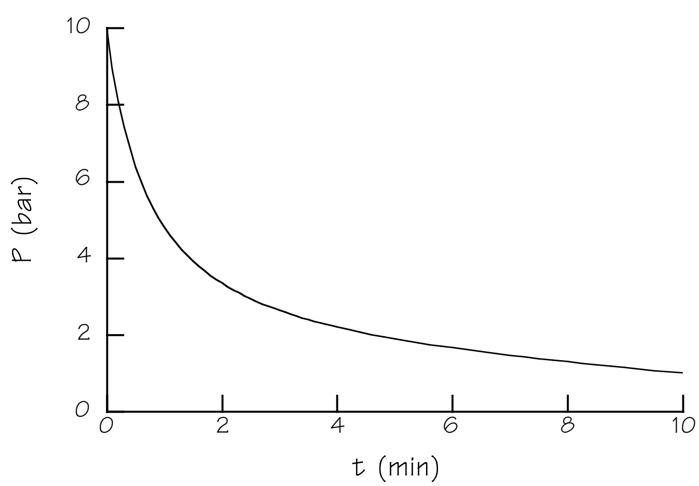



***Figure T1.5*** Pressure in the ruptured tank (Figure T1.4) as the contents discharge into the surroundings. Curve computed from (T1.36).

## *Problems*

T1.1 Consider again the tank of compressed air, introduced in § T1.3. The tank is initially charged to 10 bar at 20°C. Instead, of a failure that creates a small hole, assume a seam of the tank suddenly gives way, instantaneously and adiabatically releasing all the pressurized air. If the entire pressure change for such a process goes to increase the kinetic energy of the released air, what is the total energy release (MJ)?

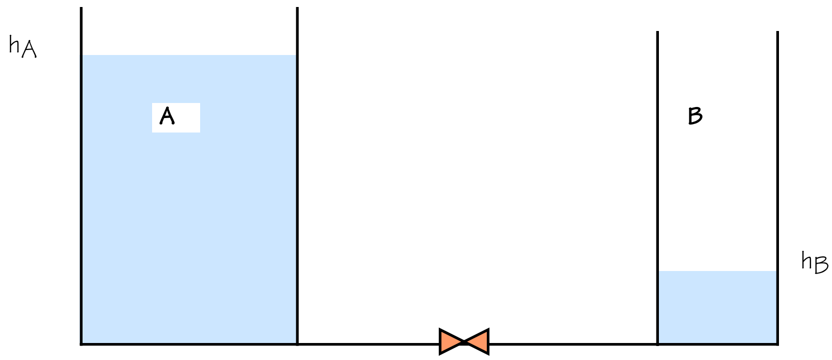
T1.2 For the ruptured-tank problem described in § T1.3, (a) determine the initial value for the linear velocity of air leaving the tank. Give your value in miles/hr. (b) Determine how the linear velocity changes with time during the discharge and make a plot of $\upsilon$ vs $t$. Is your result linear?

T1.3 A cylindrical water tank is 2 ft in diameter, 6 ft tall, and the top is open to the atmosphere. Water can be pumped into the tank, at the top, at a constant rate of 5.6 $lb_m$/s. A 2-in (ID) horizontal drain line leads from the bottom of the tank, through a valve, to an open drain. Frictional losses through the line and the open valve can be represented by (T1.11) and (T1.12) with $C = 0.75$. (a) When steady-state flow is established, what is the height of water in the tank? (b) Initially, the tank is dry, then the pump is turned on to start feeding water to the tank. How long is required for the level in the tank to reach 95% of the steady-state height? To reach 99%?

T1.4 A cylindrical tank, with diameter = 4 ft and height of 8 feet, is filled to a height of 5 ft with water. Valves in inlet and outlet lines are all closed; however, there appears to be a leak. Over 24 hours, the level in the tank (observed by sight glass) has fallen by 1 in. The flow rate is so small that frictional losses can be ignored. If the leak is a circular hole in the bottom of the tank, what is the diameter of the hole? Is your value for the diameter an upper or a lower bound—that is, if the hole were in the tank wall above the bottom, would the hole be larger or smaller than your value?

T1.5 Two open-top tanks are connected by a horizontal line at their bottoms, as shown in the following figure. The connecting pipe has an ID of 2 in. Initially, the connecting valve is closed, tank A is filled to a height $h_o$ with water, and tank B is dry.

(a) Show that, when the valve is opened and equilibrium is re-established, the final height of water in each tank is given by $h_f = h_o/\xi$, where $\xi$ is related to the cross-sectional areas of the two tanks by $\xi = 1 + \mathcal{A}_B/\mathcal{A}_A$.




(b) Perform a steady-state analysis, as in § T1.1, to show that the linear velocity of water entering tank B can be modeled as

$$\upsilon = k\sqrt{\Delta h} \qquad \text{where} \qquad k = \sqrt{2(1-C)g}$$

and $\Delta h = h_A - h_B$ is the difference in water levels while $C$ is the fraction of the head needed to overcome friction.

(c) Now show that the difference in levels changes with time according to

$$\sqrt{\Delta h} - \sqrt{\Delta h} = -\frac{1}{2}\xi\beta^2 kt$$

where $\beta = d_p/d_B$, is the ratio of pipe diameter to tank B diameter.

(d) The minimum time for the two levels to equalize occurs when there is negligible friction ($C = 0$). For $\Delta h_o = 10$ ft, $d_p = 2$ in, $d_B = 5$ ft, determine the minimum time to equalize if tank A is the same diameter as B. Determine the minimum time if tank A has twice the diameter as B. Ditto for tank A half the diameter as B.