One-Dimensional Domino Effect

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In this document we report results from simulations of the domino effect in one dimension. The basic game proceeds like this: along a one-dimensional line we attempt to stand dominoes, all of the same size. At intervals we start an “event” by pushing over one domino, causing a cascade of other dominoes to fall. For a line of fixed length and dominoes of uniform fixed dimensions, we seek the probability distribution $P(N)$ that N dominoes will fall during one event.

1. Basic Game

We choose the unit of length to be the height of each domino; i.e., $h = 1$ unit. Then each domino has thickness t units and the one-dimensional line is of length L units. When the dominoes are closed packed, with the center of the first domino at one end of the line, we obtain the maximum number of dominoes that can occupy the line,

$$N_{max} = \frac{L}{t} + 1 \tag{1}$$

In each “move” of the game, we attempt to add a domino to the line by throwing a random position, uniforming distributed on $[0, L]$. If that position is already occupied, the move is rejected. If it is unoccupied, we add the domino to the line. The position p_i of each domino i is measured relative to the center of its base; thus, domino i occupies the space from $(p_i - t/2)$ to $(p_i + t/2)$ on the line. See Figure 1.

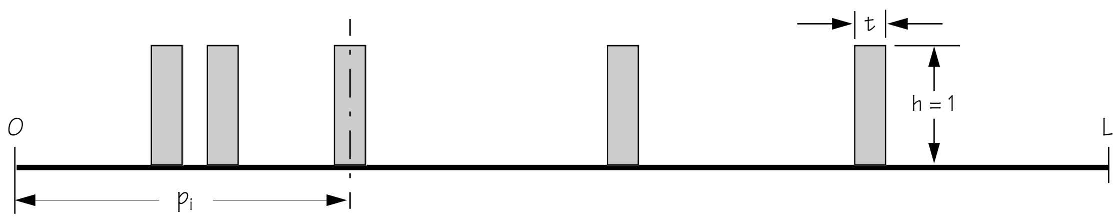

Figure 1. Basic geometry of the one-dimensional domino game.

After M moves we cause an event by randomly selecting one domino on the line and pushing it over. We flip a coin to decide whether the push is to the right or to the left. We then count the number of dominoes that fall during the event. When domino i falls, it will knock over domino j provided the horizontal distance d between their centers obeys

$$t < d = p_j - p_i < h + t \qquad \text{for falls to right} \qquad (2)$$

or

$$t < d = p_i - p_j < h + t \qquad \text{for falls to left} \qquad (3)$$

At the end of each event, we remove from the line all dominoes that have fallen; this creates a "clearing" on the line in which new dominoes can be easily added. The number of moves M between successive events is not fixed; rather, it is selected randomly from a uniform distribution on [1, M_{max}]. We start a game with an empty line and throw a random value for M to determine when the first event will occur.

The probability that N fall during one event is the frequency of observing N,

$$P(N) = \frac{E(N)}{E_{total}} \qquad (4)$$

where $E(N)$ is the number of events observed to have N dominoes fall and E_{total} is the total number of events during one simulation. We find that the behavior of the distribution P(N) is affected by the maximum number of moves M_{max} allowed between events.

2. Sample Results

Figure 2 shows sample results for $P(N)$ obtained from a simulation using dominoes of thickness $t = 0.1$ on a line of length $L = 50$. This simulation extended over 6×10^6 moves, with events occurring at random intervals of M moves, with $M_{max} = 500$. For random intervals M uniformly distributed on [1, 500], the average number of moves between events would be $500/2 = 250$. So we expect to see about $6(10^6)/250 = 24{,}000$ events. The number actually observed in this simulation was 23,817 events. The average number of dominoes to fall in one event was found to be 93.1, while the plot in Figure 2 shows that $N = 1$ is the most probable number of dominoes to fall.

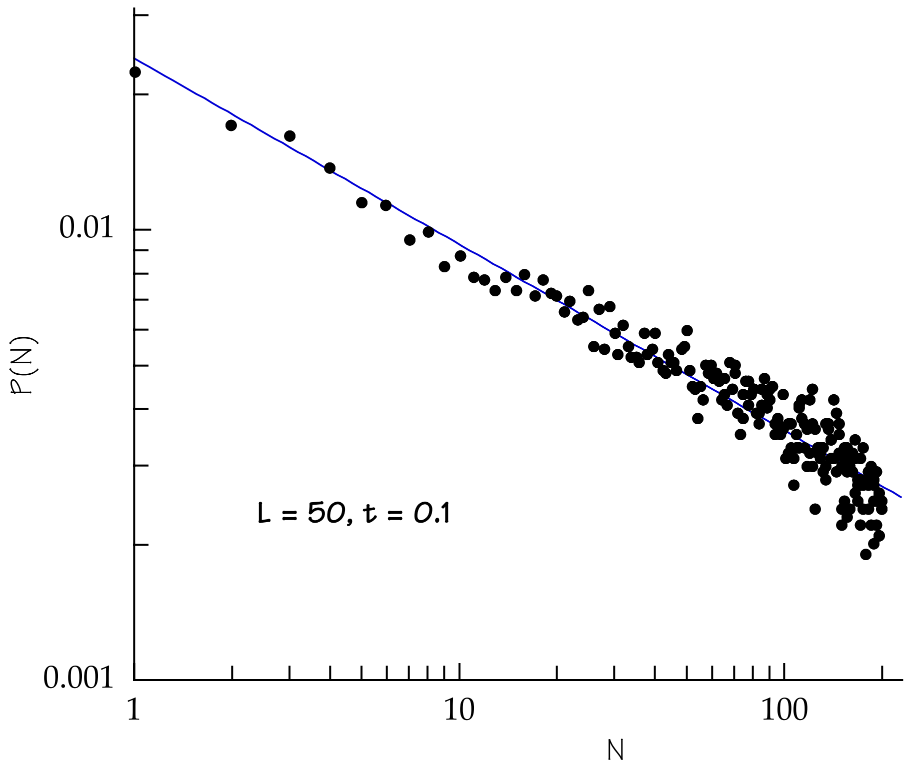

Figure 2. Simulation results (points) for probability $P(N)$ that N dominoes will fall during one event along a line of length $L = 50$ using dominoes of thickness $t = 0.1$ and height = 1. The simulation extended over 6×10^6 moves with events selected randomly from a uniform distribution on [1, 500] moves. Straight line is a least-squares fit to a power law.

This large difference between the "average" event ($N = 93$) and the most probable event ($N = 1$) implies that there is no "typical" event; or, to turn this around, any event having $N < 200$ is typical of all the others. An event having $N = 2$ is essentially the same as one having $N = 20$ and both are essentially the same as one having $N = 200$. A "catastrophe" having $N = 200$ has the same cause as a minor event having $N = 2$: each is started by a single domino falling. Nothing big, special, or unique starts a catastrophe; whether a catastrophe occurs depends on how the system is organized—how the dominoes stand relative to one another. Such scale-invariant behavior is characteristic of systems that display power-law behavior, as in Figure 2.

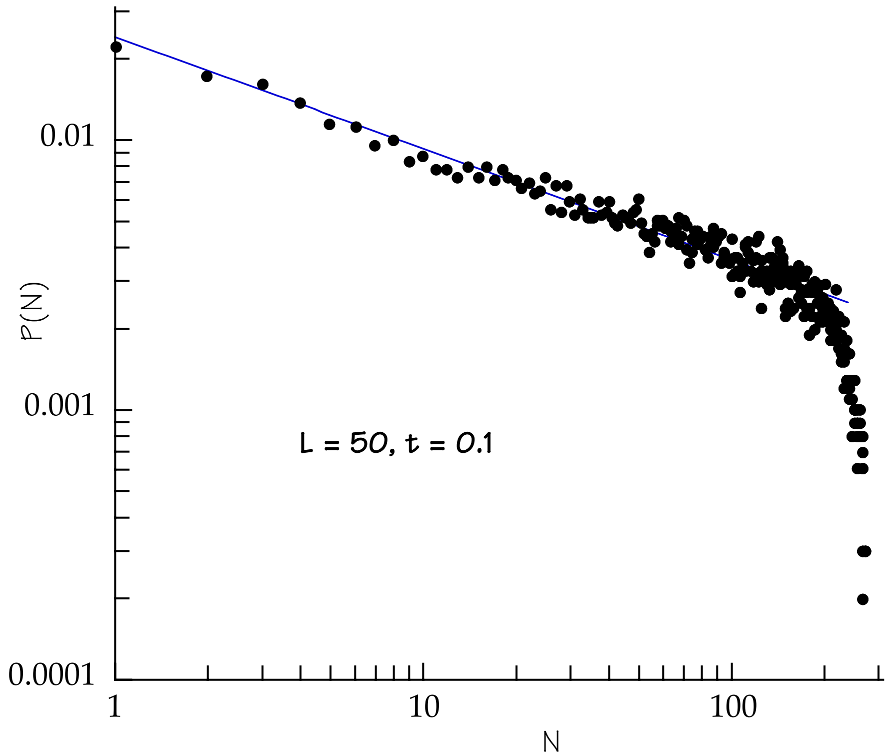

Figure 3. Simulation results from Figure 2 extended to $N = 275$ dominoes falling in one event.

3. Excluded-Space Effect

For the geometry of this simulation, the maximum number of dominoes that could occupy the line is $N_{max} = (50/0.1 + 1) = 501$, but events having $N > 200$ do not obey the power law of Figure 2. Instead, as shown in Figure 3, the probability decreases sharply for $N > 200$. This is caused by an excluded-space effect: it becomes difficult to randomly load many more than 200 dominoes of thickness $t = 0.1$ onto a line of length $L = 50$, because much of the unoccupied space is excluded to additional dominoes. Note that the excluded space is larger than simply the space actually occupied by N dominoes. For example, if one domino is well separated from the others, then that domino excludes from the center of another domino a space of length equal to two thicknesses ($2t$), as shown in the middle of Figure 4. But if that domino is close to another, so the two centers are separated by a distance $d < 2t$, then the excluded spaces of the two dominoes overlap. Now the total excluded space is greater than $2t$ but less than $4t$, as on the right in Figure 4.

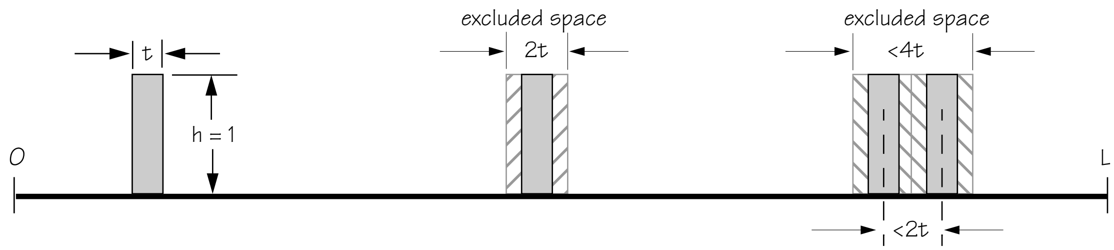

Figure 4. A domino well separated from any other domino (center) occupies a distance t on the line, but it prevents another domino's center from occupying any position within $\pm t$ of its center: the excluded distance is $2t$. But when two dominoes have centers less than $2t$ apart (right), then the space excluded to a third domino is more than the space ($2t$) actually occupied by the two, but less than the total space ($4t$) that would be excluded if the centers were more than $2t$ apart.

In general, for a line of fixed length, we can say that once the number of dominoes on the line exceeds about half the maximum possible value, then the excluded space becomes a complicated function of the number of dominoes present. In fact, once N is just a few more than $N_{max}/2$, it becomes geometrically possible for the dominoes to be separated such that all the unoccupied space on the line is excluded and no more dominoes can be added to the line. And so, although we can, by close packing, place 501 dominoes on the line, in reality we have little chance of randomly loading (say) 300 dominoes onto the line; consequently, we will have even less chance for knocking down 300 dominoes in one event.

4. Effect of Interval Between Events

Aside from the basic geometry of the situation, another parameter that may influence results for the probability $P(N)$ is M_{max}, the maximum number of moves allowed between events. We expect that as M_{max} is increased, more opportunities are made available to add dominoes to the line, thereby increasing the probabilities for large numbers of dominoes to fall. In fact, the observed results are more subtle.

Figure 5 shows results for $P(N)$ from four simulations, all for dominoes of thickness $t = 0.25$ on a line of length $L = 10$. Now the maximum number of dominoes that could occupy the line would be $N_{max} = (L/t + 1) = 41$. Each run extended to 8×10^6 moves, except that for $M_{max} = 1200$, which was for 12×10^6 moves; otherwise, the only difference in input to the runs was a systematic increase in M_{max} from 50 moves to 1200 moves. The figure shows these trends:

1. As M_{max} is increased, $P(N)$ decreases for small N; that is, minor events involving just a few dominoes become less likely to occur.

2. The transition to low $P(N)$ values, occurring at large N because of the excluded-space effect, sharpens as M_{max} is increased.

3. At very large M_{max} values, with $N < 20$, $P(N)$ has collapsed from a power law to a uniform distribution; that is, events having $1 \le N \le 20$ are all equally likely to occur.

The principal lesson from Figure 5 is that if we want to reduce the possibilities for large catastrophic events, we should initiate events at relatively short intervals. Then most events will involve only a few dominoes and opportunities for building up stands of large numbers of dominoes will be reduced.

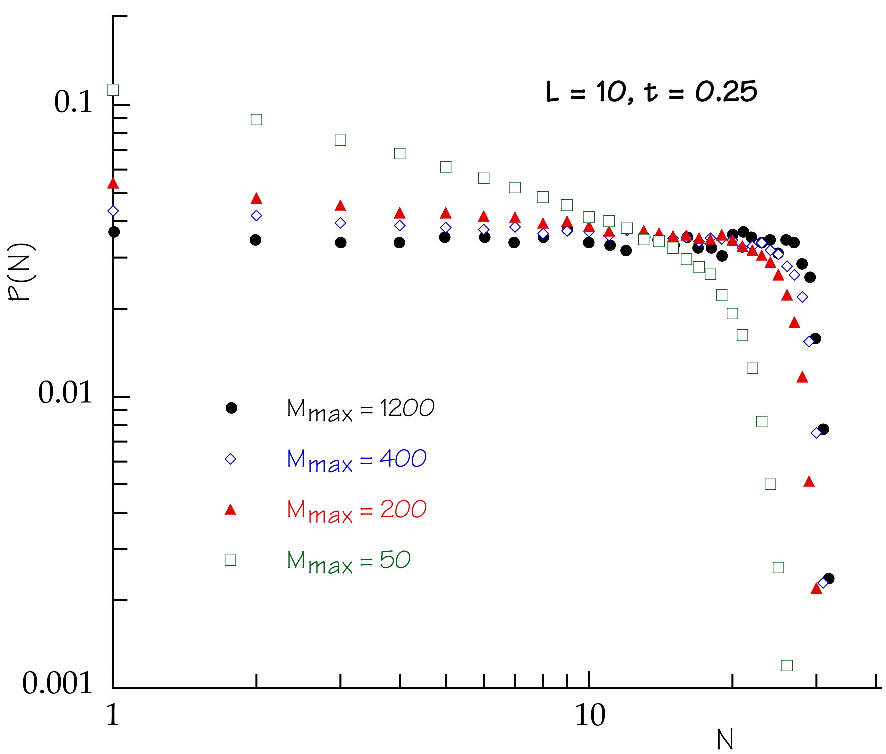

Figure 5. Results for $P(N)$ from four simulations, all using dominoes of height = 1 and thickness = 0.25 on a line of length $L = 10$. M_{max} is the maximum number of moves allowed between two events. The maximum number of dominoes that could occupy the line is $(L/t + 1) = 41$.

5. Effects of Changing the Ratios *L*/*h*, *L*/*t*, and *t*/*h*

The geometry of our one-dimensional situation is determined by three lengths: the length of the line L, the height of a domino h and its thickness t. From these we can form three dimensionless parameters: L/t, L/h, and t/h. Then only two of these three are independent. Therefore, we can fix any one ratio and perform simulations to study how $P(N)$ responds to systematic changes in a second ratio. Assume we want to know how to reduce the possibilities of having catastrophic events in which a large number of dominoes fall; or equivalently, we want to favor small-N events over large-N ones. To address this, we study from simulations the probability that a single domino falls in an event; we denote this probability as $P(N{=}1)$.

Response to changes in *L*/*h* at fixed *L*/*t*. According to (1), by fixing the ratio L/t, we fix the maximum number of dominoes that can be close-packed on the line. Nevertheless, changing L/h affects $P(N)$. When L/h is small, there is a higher likelihood that one falling domino will knock down more than one of its neighbors. Then the average number of dominoes falling per event will be relatively high, while the probability of a single domino falling will be low. As the length of the line is increased relative to h, the average number of dominoes falling per event decreases, and the probability of a single domino falling increases, even though L/t remains the same. These trends are shown as the open squares in Figure 6.

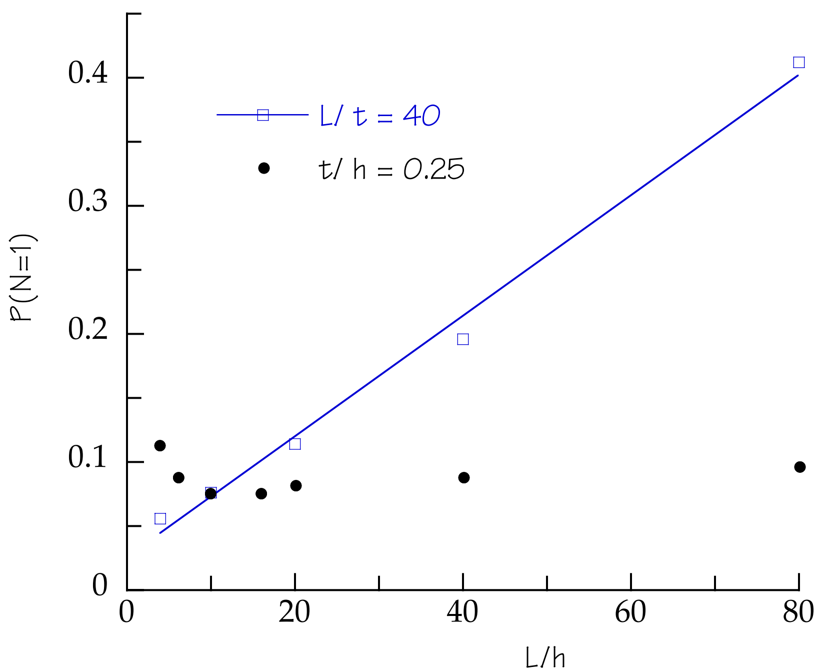

Figure 6. Simulation results for effects of L/h on probability that only one domino falls during an event. Open squares are at fixed $L/t = 40$; straight line is a linear least-squares fit to the data. Filled circles are at fixed $t/h = 0.25$. All runs used $M_{max} = 100$ and extended to 10^6 moves.

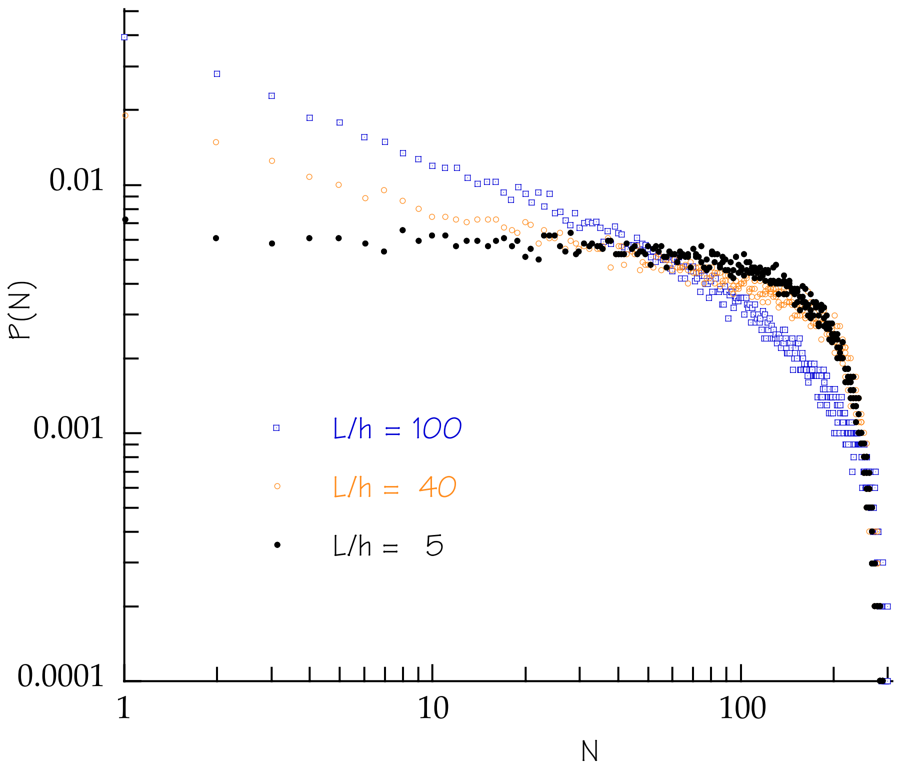

Figure 7. Response of $P(N)$ to changes in L/h from simulations at fixed $L/t = 500$. All runs used M_{max} = 500 and extended over 16(10^6) moves.

We also ask how changing L/h affects the transition to low $P(N)$ values at large N. Simulation results in Figure 7 show that decreasing L/h, with L/t fixed, sharpens the transition and collapses the $P(N)$ curve toward a uniform distribution.

Response to changes in *L/h* at fixed *t/h*. Now we ask how $P(N=1)$ responds when we fix the height and thickness of each domino while increasing the length of the line. Increasing L allows us to add more dominoes to the line, but if M_{max} (the maximum number of allowed moves between events) is also fixed, then we expect the average spacing between dominoes will also increase, thereby increasing the probability $P(N=1)$ that only one domino falls in an event. The filled circles in Figure 6 show this expected behavior, although the increase in $P(N=1)$ is much weaker than that found when we fix L/t rather than t/h.

More interesting is that the curve formed by the filled circles in Figure 6 passes through a weak minimum at small L/h values. When t/h fixed and L/h is very small, then only one or two dominoes can occupy the line, and the probability is large that only one falls during an event. But as

L/h is increased, and more dominoes can occupy the line, $P(N=1)$ falls rapidly, before it begins to increase, as we see in Figure 6.

Response to changes in *L/t* at fixed *L/h*. Finally, we consider how $P(N=1)$ responds when we fix L/h and change L/t; by (1), manipulating L/t changes the maximum number of dominoes that can be close-packed onto the line. As L/t is increased, the maximum (and average) number of dominoes on the line increases, tending to redistribute the probabilities in favor of large numbers of dominoes falling during an event at the expense of few dominoes falling. This expected behavior is found in the simulation results plotted in Figure 8 (open triangles).

Also shown, for comparison, in Figure 8 is the response to changes in L/t at fixed t/h (filled circles); these are the same data as the filled circles in Figure 6. When L/t increases at fixed t/h, the ratio L/t also increases, although the weak minimum in $P(N=1)$ in Figure 6 is more pronounced on the plot in Figure 8 (because of the change in scale from Figure 6 to Figure 8 and the change in variable plotted on the abscissa).

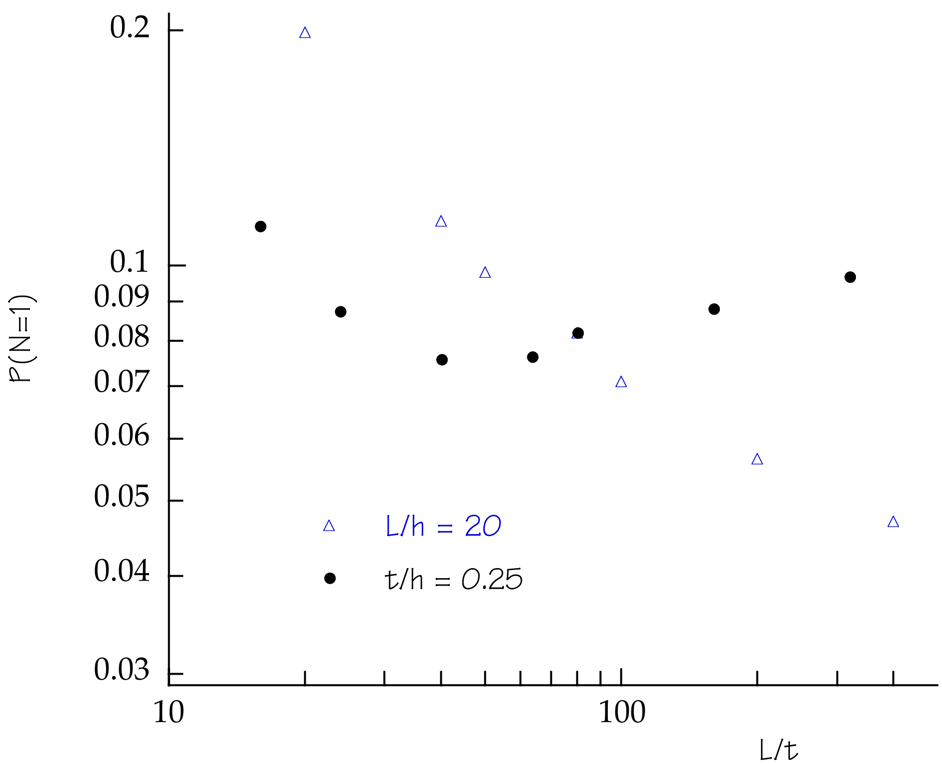

Figure 8. Simulation results for effects of L/t on probability that only one domino falls during an event. Open triangles are at fixed $L/h = 20$. Filled circles are at fixed $t/h = 0.25$, and are the same data as the filled circles in Figure 6. All runs used $M_{max} = 100$ and extended to 10^6 moves.

6. Summary

In many physical situations that bear some resemblance to our one-dimensional domino game, an objective is to find ways to reduce the probabilities for large-scale catastrophes, or at least bias the behavior in favor of small-scale events. In this study we have considered the effects of four variables: the maximum allowed interval between events M_{max}, the length of the line L, the height of each domino h, and the thickness of each t. To favor small-N events over large-scale catastrophes we have the following possibilities:

1. Initiate events at relatively short intervals; i.e., decrease M_{max} (Figure 5). This leaves fewer opportunities for stands of large numbers of dominoes to accumulate.

2. If L/t is fixed, then increase L/h (Figure 6). Fixing L/t fixes the maximum (and average) number of dominoes that can occupy the line. So increasing L/h effectively spreads those dominoes farther apart (on average), leading to higher probabilities for events involving few dominoes.

3. If t/h is fixed, then increasing L/h (Figure 6) will tend to favor small-N events. If t and h are individually fixed, then the size of each domino is fixed, and the strategy is to increase L, but increasing L also increases the number of dominoes that can occupy the line, so the net effect may be small.

4. If L/h is fixed, then decrease L/t (Figure 8). Decreasing L/t reduces the maximum (and average) number of dominoes that can occupy the line, spreading the remaining ones farther apart (on average), leading to higher probabilities for small-N events.