Dialog on Simple Derivatives

Bette: Excuse me, Prof, could Alf and I talk to you a few minutes?

Prof: Oh Hi, Bette. Sure. What's the problem?

Alf: We're having problems with these total and partial derivative things.

Bette: Yeah, we just don't seem to understand what they're about.

Prof: You have had the calculus sequence?

Alf: Yeah, but this doesn't seem to be the same.

Prof: Are you telling me that calculus in an engineering building differs from calculus in a math building?

Bette: Well, in the calculus courses we just had to manipulate terms and equations, but now I think you're really expecting us to understand what's going on.

Prof: Indeed.

Alf: We need help, Prof.

Prof: OK, you've come to the right shop. Let's start with a circle. Could you please draw us a circle on the board, Alf?

Alf: This looks a little lopsided.

Prof: It'll serve. So, Bette, if we wanted the area of this circle, how would you find it?

Bette: $A = \pi r^2$

Prof: Good. So the area depends on only one variable—the radius? If you know the radius, then you can compute the area?

Alf: Sure Prof.

Prof: What about the diameter? Does the area depend on the diameter?

Bette: Well, we could write the formula as $A = \pi d^2/4$.

Alf: But if we know the radius, we also know the diameter.

Prof: So either way, the area depends on only one variable, and we can choose whether that one variable is the radius or the diameter or something else?

Alf: Right.

Prof: OK. So, what's the derivative of the area wrt the radius?

Bette: It's $dA/dr = 2\pi r$.

Prof: Good. Now, Bette, what does this derivative mean? What does it tell us?

Bette: I'm not sure—it's something about a slope.

Alf: No, Bette, it is a slope. The derivative is a slope.

Prof: OK, Alf, slope of what?

Alf: Gee, I'm not sure.

Prof: Look—the notation is designed to tell you. Our expression for the derivative is this

$$\frac{dA}{dr} = 2\pi r$$

Let's make a plot. Look at the derivative on the left of the equal mark. The quantity on top is called the dependent variable; here it's the area A. So let's plot A on an ordinate. The quantity on the bottom is the independent variable—it's the radius r. Let's plot that on an abscissa. So let's make a plot of A vs r. What would it look like?

Alf: I'm not sure.

Prof: OK, Bette, what did you say is the relation between A and r?

Bette: $A = \pi r^2$

Prof: OK, Alf, if we plot A vs r, what kind of curve do we get—a straight line?

Alf: No, Prof, it's quadratic.

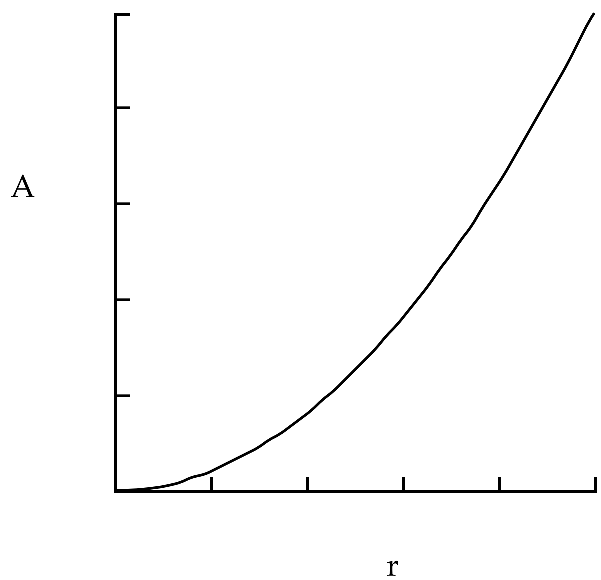
Prof: OK, do we know any point on the quadratic?

Alf: Well, we could pick a radius—

Prof: OK, say we pick r = 0. Then what?

Bette: Then A = 0.

Prof: OK, so our curve is a quadratic through the origin. It looks like this:



Prof: Now, where's the derivative on this plot?

Alf: Ah, it's the slope of the curve.

Prof: Good. But the slope at which point on the curve?

Bette: It's the slope at every point.

Prof: So the slope isn't constant?

Alf: No, our equation is slope = $2\pi r$, so it must change with r.

Prof: Good. If the slope were constant, what kind of curve would we have?

Bette: A straight line.

Prof: Good. So we have this interpretation of a derivative as a slope of a curve. But what does this mean physically?

Alf: I don't get the question.

Prof: OK, let me ask it this way: How did you know that $dA/dr = 2\pi r$?

Bette: We learned that in calculus.

Prof: OK, but where does the formula for the derivative of a power ($dr^2/dr = 2r$) come from?

Alf: It's in the calculus book, Prof.

Prof: OK. But your motivation, if I recall, is to get beyond books to understanding.

Bette: That's right.

Prof: So you must understand that a derivative tells us how a quantity (the dependent variable) responds when we change another quantity (the independent variable). For a circle, the derivative dA/dr tells us how the area changes when we manipulate the radius. For example, does the area increase or decrease when we increase the radius?

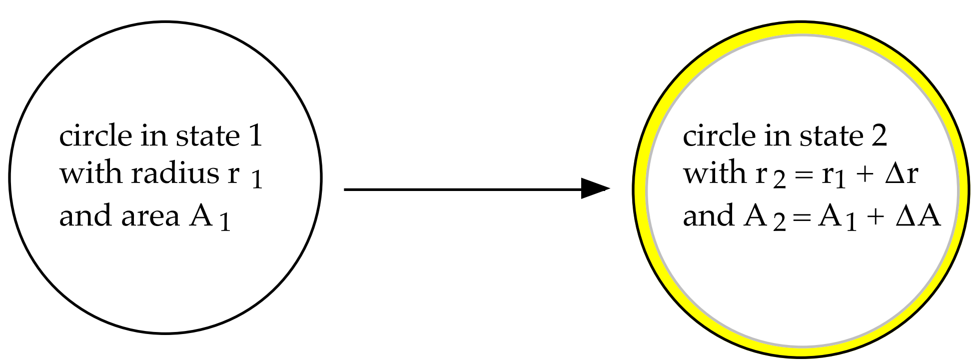
Alf: It increases.

Prof: And is the derivative dA/dr positive or negative?

Bette: $dA/dr = 2\pi r > 0$. It's always positive.

Alf: But you haven't said where the formula comes from.

Prof: OK, lets consider a circle that initially has a radius r_1 and area A_1. Then we want to increase the radius by a small amount Δr to a new value r_2. So we have a process like this:



Prof: The shading in state 2 indicates the thickness Δr. The question is, How does the area change? Symbolically, we can write $A_2 = A_1 + \Delta A$, but what would be the value for ΔA?

Alf: It will depend on the size of the change Δr.

Prof: OK, but can we be quantitative? What's the equation for the new area?

Bette: $A_2 = \pi r_2^2$

Prof: Now, let's write this in terms of A_1 and r_1, so

$$A_2 = \pi r_2^2$$

becomes

$$A_1 + \Delta A = \pi (r_1 + \Delta r)^2$$

Now write A_1 in terms of r_1,

$$\pi r_1^2 + \Delta A = \pi (r_1 + \Delta r)^2$$

Expand the quadratic on the rhs,

$$\pi r_1^2 + \Delta A = \pi [r_1^2 + 2r\Delta r + (\Delta r)^2]$$

Prof: Cancel the common term from lhs and rhs,

$$\Delta A = \pi [2r\Delta r + (\Delta r)^2]$$

and form the ratio $\Delta A/\Delta r$,

$$\frac{\Delta A}{\Delta r} = \pi (2r + \Delta r)$$

Now, can we get the derivative from this?

Alf: I think there's something about a limit.

Prof: Yes. You see, so far we have been considering a finite change in the radius; this is what we mean by the notation Δr. For example, if the circle initially had a radius $r_1 = 6$ inches, then the increase in r might be, say, $\Delta r = 1/2$-inch.

Bette: So, what's wrong with that?

Prof: Nothing, in terms of the physical situation. But remember our curve of A vs r? The value of the slope differs at each r; so, if we want a particular value for the slope, we must evaluate the slope at a particular point r. Yet, in our analysis thus far, we have considered a range of r-values; this is represented by Δr.

Alf: So we need to make Δr small?

Prof: Exactly, Alf. We must shrink the increment Δr down to nearly zero; this process is represented by the limit. Therefore, our ratio of finite differences, $\Delta A/\Delta r$, becomes the derivative in the limit that Δr approaches zero,

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = 2\pi\, r$$

Bette: Say—that's what we wrote for the derivative originally.

Prof: Exactly the point, Bette. We've derived the expression that you told me you found in your calculus book.

Alf: Let me get this straight, Prof. The derivative dA/dr tells us how the area A changes when we change the radius r by a very small amount. For a circle, the derivative is not constant, but depends on the initial value of the radius.

Prof: You've got it, Alf.

Alf: All right! But wait a minute, Prof, why are you writing dA/dr instead of $\partial A/\partial r$ like you do in class?

Prof: The notation dA/dr indicates the total derivative; $\partial A/\partial r$ would indicate a partial derivative.

Bette: But what's the difference? That's what we came here to find out.

Prof: The total derivative represents the response of A to the manipulations of all independent variables. Since the area of a circle depends on only one independent variable (say, r), the total derivative is the same as the simple derivative we've been discussing. Partial derivatives occur when a variable can be changed by manipulating more than one independent variable. For the area of a circle, there are no additional independent variables, so there's no need to use the notation for a partial derivative.

Alf: I'm still not clear, Prof.

Prof: Well, I think we should leave that for another day. I'd prefer you to solidify you're understanding of the derivative of a function of one variable. Then we can discuss partial derivatives.

Bette: Are there any thermodynamic properties that depend on only one variable?

Prof: A few, Bette, like the second virial coefficient for a pure gas. It only depends on temperature, so we write dB/dT, and not ∂B/∂T.

Alf: You said solidify, Prof?

Prof: OK, Alf, pick some other quantities that depend on only one variable—like the volume of a sphere, the perimeter of a square, the hypotenuse of a right isosceles triangle—and do what we just did. In particular, (1) make the plot that connects slope to variables and (2) do the derivation of the formula for the derivative from the finite difference analysis.

Bette: OK, Prof, we'll be back in a couple of days to talk about partial derivatives.

Prof: Good.